

A Note on Large Deviations for 2D Coulomb Gas with Weakly Confining Potential

Adrien Hardy ^{*}†

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Abstract

We investigate a Coulomb gas in a potential satisfying a weaker growth assumption than usual and establish a large deviation principle for its empirical measure. As a consequence the empirical measure is seen to converge towards a non-random limiting measure, characterized by a variational principle from logarithmic potential theory, which may not have compact support. The proof of the large deviation upper bound is based on a compactification procedure which may be of help for further large deviation principles.

1 Introduction and statement of the result

Consider the distribution of N real particles x_1, \dots, x_N which interact like a 2D Coulomb gas at inverse temperature $\beta > 0$ under an external potential. Namely, let \mathbb{P}_N be the probability distribution on \mathbb{R}^N with density

$$\frac{1}{Z_N} \prod_{1 \leq i < j \leq N} |x_i - x_j|^\beta \prod_{i=1}^N e^{-NV(x_i)}, \quad (1.1)$$

where the so-called potential $V : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function growing sufficiently fast as $|x| \rightarrow \infty$ so that

$$Z_N = \int \cdots \int_{\mathbb{R}^N} \prod_{1 \leq i < j \leq N} |x_i - x_j|^\beta \prod_{i=1}^N e^{-NV(x_i)} dx_i < +\infty. \quad (1.2)$$

For $\beta = 1$ (resp. $\beta = 2$ and 4) such a density is known to match with the joint eigenvalue distribution of a $N \times N$ orthogonal (resp. unitary and unitary symplectic) invariant Hermitian random matrix [DG09]. In this work, our interest lies in the limiting global distribution of the x_i 's as $N \rightarrow \infty$, that is the convergence of the empirical measure

$$\mu^N = \frac{1}{N} \sum_{i=1}^N \delta_{x_i}. \quad (1.3)$$

^{*}Institut de Mathématiques de Toulouse, Université de Toulouse, 31062 Toulouse, France.

[†]Department of Mathematics, Katholieke Universiteit Leuven, Celestijnenlaan 200 B, 3001 Leuven, Belgium. Email address: adrien.hardy@wis.kuleuven.be

Note the μ^N 's are random variables taking their values in the space $\mathcal{M}_1(\mathbb{R})$ of probability measures on \mathbb{R} , that we equip with the usual weak topology. The almost sure convergence of $(\mu^N)_N$ towards a non-random limit μ_V^* is classically known to hold under the hypothesis that there exists $\beta' > 1$ satisfying $\beta' \geq \beta$ such that

$$\lim_{|x| \rightarrow \infty} \left\{ V(x) - \beta' \log |x| \right\} = +\infty, \quad (1.4)$$

that is, as $|x| \rightarrow \infty$, the confinement effect due to the potential V is stronger than the repulsion between the x_i 's. The limiting distribution μ_V^* is then characterized as the unique minimizer of the functional

$$I_V(\mu) = \iint F_V(x, y) d\mu(x) d\mu(y), \quad \mu \in \mathcal{M}_1(\mathbb{R}), \quad (1.5)$$

where we introduced the following variation of the weighted logarithmic kernel

$$F_V(x, y) = \frac{\beta}{2} \log \frac{1}{|x - y|} + \frac{1}{2} V(x) + \frac{1}{2} V(y), \quad x, y \in \mathbb{R}. \quad (1.6)$$

A stronger statement, first established by Ben Arous and Guionnet for a Gaussian potential $V(x) = x^2/2$ [BAG97] and later extended to arbitrary continuous potential V satisfying the growth condition (1.4) [HP00], [AGZ10], is that $(\mu^N)_N$ satisfies a large deviation principle (LDP) on $\mathcal{M}_1(\mathbb{R})$ in the scale N^2 and good rate function $I_V - I_V(\mu_V^*)$. It is moreover known that μ_V^* has a compact support [AGZ10, Lemma 2.6.2].

It is the aim of this work to show that such statements still hold, except that μ_V^* may not have compact support, when one allows the confining effect of the potential V to be of the same order of magnitude than the repulsion between the x_i 's. Namely, we consider the following weaker growth condition: there exists $\beta' > 1$ satisfying $\beta' \geq \beta$ such that

$$\liminf_{|x| \rightarrow \infty} \left\{ V(x) - \beta' \log |x| \right\} > -\infty. \quad (1.7)$$

More precisely, we will establish the following.

Theorem 1.1. *Under the growth assumption (1.7),*

- (a) *The level set $\{\mu \in \mathcal{M}_1(\mathbb{R}) : I_V(\mu) \leq \alpha\}$ is compact for any $\alpha \in \mathbb{R}$.*
- (b) *I_V admits a unique minimizer μ_V^* on $\mathcal{M}_1(\mathbb{R})$.*
- (c) *For any closed set $\mathcal{F} \subset \mathcal{M}_1(\mathbb{R})$,*

$$\limsup_{N \rightarrow \infty} \frac{1}{N^2} \log \mathbb{P}_N(\mu^N \in \mathcal{F}) \leq - \inf_{\mu \in \mathcal{F}} \left\{ I_V(\mu) - I_V(\mu_V^*) \right\}.$$

- (d) *For any open set $\mathcal{O} \subset \mathcal{M}_1(\mathbb{R})$,*

$$\liminf_{N \rightarrow \infty} \frac{1}{N^2} \log \mathbb{P}_N(\mu^N \in \mathcal{O}) \geq - \inf_{\mu \in \mathcal{O}} \left\{ I_V(\mu) - I_V(\mu_V^*) \right\}.$$

Note that (1.7), together with the inequality $|x - y| \leq (1 + |x|)(1 + |y|)$, $x, y \in \mathbb{R}$, yields (1.2) and that F_V is bounded from below, so that I_V is well defined on $\mathcal{M}_1(\mathbb{R})$.

A consequence of Theorem 1.1 (b) and (c), together with the Borel-Cantelli Lemma, is the almost sure convergence of $(\mu^N)_N$ towards μ_V^* in the weak topology of $\mathcal{M}_1(\mathbb{R})$. Namely,

Corollary 1.2. *For any neighborhood $\mathcal{B} \subset \mathcal{M}_1(\mathbb{R})$ of μ_V^* ,*

$$\lim_{N \rightarrow \infty} \mathbb{P}_N(\mu^N \notin \mathcal{B}) = 0.$$

Let us now give a simple example arising from random matrix theory where the limiting distribution μ_V^* has unbounded support.

Example 1.3. On the space $\mathcal{H}_N(\mathbb{C})$ of $N \times N$ Hermitian complex matrices, consider the probability distribution

$$\frac{1}{Z_N} \det(I_N + X^2)^{-N} dX,$$

where $I_N \in \mathcal{H}_N$ is the identity matrix, dX the Lebesgue measure of $\mathcal{H}_N \simeq \mathbb{R}^{N^2}$ and Z_N a normalization constant. Performing a spectral decomposition and integrating out the eigenvectors, it is known that the induced distribution for the eigenvalues is given by (1.1) with $\beta = 2$, $V(x) = \log(1 + x^2)$, and some new normalization constant Z_N . One can then compute (see also Remark 2.2) that the minimizer of (1.5) is the Cauchy distribution

$$d\mu_V^*(x) = \frac{1}{\pi(1 + x^2)} dx. \quad (1.8)$$

Remark 1.4. (Exponential tightness and compactification)

The proof of Theorem 1.1 under the stronger growth assumption (1.4) presented in [BAG97],[HP00],[AGZ10] follows a classical strategy in LDPs theory (see e.g [DZ98] for an introduction), that is to control the deviations of $(\mu^N)_N$ towards arbitrary small balls of $\mathcal{M}_1(\mathbb{R})$, and then prove an exponential tightness property for $(\mu^N)_N$: There exists a sequence of compact sets $(\mathcal{K}_L)_L \subset \mathcal{M}_1(\mathbb{R})$ such that

$$\limsup_{L \rightarrow \infty} \limsup_{N \rightarrow \infty} \frac{1}{N^2} \log \mathbb{P}_N(\mu^N \notin \mathcal{K}_L) = -\infty. \quad (1.9)$$

Under the weaker growth assumption (1.7), it is not clear to the author how to prove the exponential tightness for $(\mu^N)_N$ directly, and we thus prove Theorem 1.1 by using a different approach. We adapt an idea of [HK11] and map \mathbb{R} on a circle \mathcal{S} , homeomorphic to the Alexandrov compactification of \mathbb{R} by the inverse stereographic projection T , then push-forward $\mathcal{M}_1(\mathbb{R})$ to $\mathcal{M}_1(\mathcal{S})$, and take advantage that the latter set is compact for its weak topology. More precisely, it will be seen that it is enough to establish upper bounds for the deviations of $(T_*\mu^N)_N$, the push-forward of $(\mu^N)_N$ by T , towards arbitrary small balls of $\mathcal{M}_1(\mathcal{S})$. The latter fact is possible thanks to the explicit change of metric induced by T .

The next section details the proof of Theorem 1.1. We first describe the announced compactification procedure in Section 2.1. Then, we study $(T_*\mu^N)_N$ and

a related rate function on $\mathcal{M}_1(\mathcal{S})$ in Section 2.2. From these informations, we are able to provide a proof for Theorem 1.1 in Section 2.3. Finally, we discuss in Section 3 some generalizations concerning the support of the Coulomb gas and the compactification procedure of possible further interest.

2 Proof of Theorem 1.1

We first describe the compactification procedure.

2.1 Compactification

Let \mathcal{S} be the circle of \mathbb{R}^2 centered in $(0, 1/2)$ of radius $1/2$ and $T : \mathbb{R} \rightarrow \mathcal{S}$ the associated inverse stereographic projection, namely the map defined by

$$T(x) = \left(\frac{x}{1+x^2}, \frac{x^2}{1+x^2} \right), \quad x \in \mathbb{R}.$$

It is known that T is a homeomorphism from \mathbb{R} onto $\mathcal{S} \setminus \{\infty\}$, where $\infty = (0, 1)$, so that (\mathcal{S}, T) is an Alexandrov compactification of \mathbb{R} . For $\mu \in \mathcal{M}_1(\mathbb{R})$, we denote by $T_*\mu$ its push-forward by T , that is the measure on \mathcal{S} characterized by

$$\int_{\mathcal{S}} f(x) dT_*\mu(x) = \int_{\mathbb{R}} f(T(x)) d\mu(x) \quad (2.1)$$

for every Borel function f on \mathcal{S} . Then the following Lemma holds.

Lemma 2.1. *T_* is an homeomorphism from $\mathcal{M}_1(\mathbb{R})$ to $\{\mu \in \mathcal{M}_1(\mathcal{S}) : \mu(\{\infty\}) = 0\}$.*

Proof. The inverse of T_* is given by push backward via T . Namely, for $\mu \in \mathcal{M}_1(\mathcal{S})$, set $T_*^{-1}\mu(A) = \mu(T(A))$ for all Borel set $A \subset \mathcal{S}$. T_*^{-1} is well-defined since $T_*^{-1}\mu(\mathbb{R}) = \mu(\mathcal{S} \setminus \{\infty\}) = 1$ as soon as $\mu(\{\infty\}) = 0$, and then T_* and T_*^{-1} are clearly reciprocal. For any continuous function f on \mathcal{S} , $f \circ T$ is continuous and bounded on \mathbb{R} , and thus T_* is continuous on $\mathcal{M}_1(\mathbb{R})$ by definition (2.1). To show the continuity of T_*^{-1} , consider a sequence $(\mu_N)_N$ in $\mathcal{M}_1(\mathcal{S})$ with weak limit μ and assume that $\mu_N(\{\infty\}) = 0$ for all N and $\mu(\{\infty\}) = 0$. Then, for any $\epsilon > 0$, the outer regularity of μ and the weak convergence of $(\mu_N)_N$ towards μ yield the existence of a neighborhood $B \subset \mathcal{S}$ of ∞ such that

$$\limsup_{N \rightarrow \infty} \mu_N(B) \leq \mu(B) \leq \epsilon,$$

and thus the sequence $(T_*^{-1}\mu_N)_N$ is tight. As a consequence, since $f \circ T^{-1}$ is continuous on \mathcal{S} for any continuous function f having compact support in \mathbb{R} , the continuity of T_*^{-1} follows. \square

The next step is to obtain an upper control on the deviation of $(T_*\mu^N)_N$ towards arbitrary small balls of $\mathcal{M}_1(\mathcal{S})$.

2.2 Weak LDP upper bound for $(T_*\mu^N)_N$

The change of metric induced by T is given by [AN07, Lemma 3.4.2]

$$|T(x) - T(y)| = \frac{|x - y|}{\sqrt{1 + x^2}\sqrt{1 + y^2}}, \quad x, y \in \mathbb{R}, \quad (2.2)$$

where $|\cdot|$ stands on both sides for the Euclidean norm. From the potential V we then construct a potential $\mathbf{V} : \mathcal{S} \rightarrow \mathbb{R} \cup \{+\infty\}$ in the following way. Set

$$\mathbf{V}(T(x)) = V(x) - \frac{\beta}{2} \log(1 + x^2), \quad x \in \mathbb{R}, \quad (2.3)$$

and

$$\mathbf{V}(\infty) = \liminf_{|x| \rightarrow \infty} \left\{ V(x) - \frac{\beta}{2} \log(1 + x^2) \right\}. \quad (2.4)$$

Note that the growth assumption (1.7) is equivalent to $\mathbf{V}(\infty) > -\infty$, so that \mathbf{V} is lower semi-continuous on \mathcal{S} . As a consequence the kernel

$$F_{\mathbf{V}}(z, w) = \frac{\beta}{2} \log \frac{1}{|z - w|} + \frac{1}{2} \mathbf{V}(z) + \frac{1}{2} \mathbf{V}(w), \quad z, w \in \mathcal{S}, \quad (2.5)$$

is lower semi-continuous and bounded from below on $\mathcal{S} \times \mathcal{S}$, and the functional

$$I_{\mathbf{V}}(\mu) = \iint F_{\mathbf{V}}(z, w) d\mu(z) d\mu(w), \quad \mu \in \mathcal{M}_1(\mathcal{S}), \quad (2.6)$$

is well-defined. The potential \mathbf{V} has been built so that the following relation holds

$$I_V(\mu) = I_{\mathbf{V}}(T_*\mu), \quad \mu \in \mathcal{M}_1(\mathbb{R}). \quad (2.7)$$

Indeed, we obtain from (2.2), (2.3) and (2.1) respectively that

$$\begin{aligned} & \iint F_V(x, y) d\mu(x) d\mu(y) \\ &= \iint \left\{ \frac{\beta}{2} \log \frac{1}{|T(x) - T(y)|} + \frac{1}{2} \mathbf{V}(T(x)) + \frac{1}{2} \mathbf{V}(T(y)) \right\} d\mu(x) d\mu(y) \\ &= \iint \left\{ \frac{\beta}{2} \log \frac{1}{|z - w|} + \frac{1}{2} \mathbf{V}(z) + \frac{1}{2} \mathbf{V}(w) \right\} dT_*\mu(z) dT_*\mu(w) \\ &= \iint F_{\mathbf{V}}(z, w) dT_*\mu(z) dT_*\mu(w). \end{aligned}$$

Let us come back to Example 1.3.

Remark 2.2. (Example 1.3, continued) For $\beta = 2$ and $V(x) = \log(1 + x^2)$, we have $\mathbf{V} = 0$ and thus from (2.7)

$$I_V(\mu) = \iint \log \frac{1}{|z - w|} dT_*\mu(z) dT_*\mu(w), \quad \mu \in \mathcal{M}_1(\mathbb{R}). \quad (2.8)$$

By rotational invariance, the minimizer of

$$\iint \log \frac{1}{|z - w|} d\nu(z) d\nu(w), \quad \nu \in \mathcal{M}_1(\mathcal{S}),$$

has to be the uniform measure $\mathcal{U}_{\mathcal{S}}$ of \mathcal{S} , and thus the minimizer μ_V^* of I_V is given by the push-backward $T_*^{-1}\mathcal{U}_{\mathcal{S}}$ which is easily seen to be the Cauchy law (1.8).

Given a metric d on $\mathcal{M}_1(\mathcal{S})$, compatible with its weak topology (such as the Lévy-Prohorov metric, see [D02]), we denote for the associated balls

$$\mathcal{B}(\mu, \delta) = \left\{ \nu \in \mathcal{M}_1(\mathcal{S}) : d(\mu, \nu) < \delta \right\}, \quad \mu \in \mathcal{M}_1(\mathcal{S}), \quad \delta > 0.$$

The following Proposition regroupes all the informations concerning $I_{\mathbf{V}}$ and $(T_*\mu^N)_N$ needed to establish Theorem 1.1 in the next Section.

Proposition 2.3.

- (a) *The level set $\{\mu \in \mathcal{M}_1(\mathcal{S}) : I_{\mathbf{V}}(\mu) \leq \alpha\}$ is closed, and thus compact, for any $\alpha \in \mathbb{R}$.*
- (b) *$I_{\mathbf{V}}$ is strictly convex on the set where it is finite.*
- (c) *For any $\mu \in \mathcal{M}_1(\mathcal{S})$, we have*

$$\limsup_{\delta \rightarrow 0} \limsup_{N \rightarrow \infty} \frac{1}{N^2} \log \left\{ Z_N \mathbb{P}_N \left(T_*\mu^N \in \mathcal{B}(\mu, \delta) \right) \right\} \leq -I_{\mathbf{V}}(\mu).$$

The proof of Proposition 2.3 is somehow classical and inspired from the ideas developed in [BAG97] (c.f. also [HP00], [AGZ10], [HK11]).

Proof. (a) It is equivalent to show that $I_{\mathbf{V}}$ is lower semi-continuous. Since $F_{\mathbf{V}}$ is lower semi-continuous, there exists an increasing sequence $(F_{\mathbf{V}}^M)_M$ of continuous functions on $\mathcal{S} \times \mathcal{S}$ satisfying $F_{\mathbf{V}} = \sup_M F_{\mathbf{V}}^M$. We obtain for any $\mu \in \mathcal{M}_1(\mathcal{S})$ by monotone convergence

$$I_{\mathbf{V}}(\mu) = \sup_M \iint F_{\mathbf{V}}^M(z, w) d\mu(z) d\mu(w),$$

and $I_{\mathbf{V}}$ is thus lower semi-continuous on $\mathcal{M}_1(\mathcal{S})$ being the supremum of a family of continuous functions.

- (b) Denote for a (possibly signed) measure μ on \mathcal{S} its logarithmic energy by

$$I(\mu) = \iint \log \frac{1}{|x - y|} d\mu(x) d\mu(y) \tag{2.9}$$

when this integral makes sense. Let $\mu, \nu \in \mathcal{M}_1(\mathcal{S})$ such that $I_{\mathbf{V}}(\mu) < +\infty$ and $I_{\mathbf{V}}(\nu) < +\infty$. In particular, because \mathbf{V} is bounded from below and \mathcal{S} is compact, they also satisfy $|I(\mu)| < +\infty$ and $|I(\nu)| < +\infty$. Then, for any $0 < t < 1$, we have

$$I_{\mathbf{V}}(t\mu + (1 - t)\nu) = t^2 I_{\mathbf{V}}(\mu) + (1 - t)^2 I_{\mathbf{V}}(\nu) - t(1 - t) I(\mu - \nu).$$

Moreover, since $I(\mu - \nu) \geq 0$ with equality if and only if $\mu = \nu$ [CKL98, Theorem 2.5], the strict convexity of $I_{\mathbf{V}}$ where it is finite follows.

- (c) Introduce for $i = 1, \dots, N$ the random variables $z_i = T(x_i)$ where the x_i 's are distributed according to (1.1) so that

$$T_*\mu^N = \frac{1}{N} \sum_{i=1}^N \delta_{z_i}. \tag{2.10}$$

We can easily compute the distribution for the z_i 's induced by (1.1). Indeed, with \mathbf{V} defined in (2.3),(2.4), we obtain from the metric relation (2.2) that

$$\begin{aligned} & \frac{1}{Z_N} \prod_{1 \leq i < j \leq N} |x_i - x_j|^\beta \prod_{i=1}^N e^{-NV(x_i)} dx_i \\ &= \frac{1}{Z_N} \prod_{1 \leq i < j \leq N} |T(x_i) - T(x_j)|^\beta \prod_{i=1}^N (1 - |T(x_i)|^2)^{\beta/2} e^{-N(V(x_i) - \frac{\beta}{2} \log(1 + |x_i|^2))} dx_i \\ &= \frac{1}{Z_N} \prod_{1 \leq i < j \leq N} |z_i - z_j|^\beta \prod_{i=1}^N (1 - |z_i|^2)^{\beta/2} e^{-N\mathbf{V}(z_i)} d\lambda(z_i), \end{aligned}$$

where λ stands for the push-forward by T of the Lebesgue measure on \mathbb{R} . As a consequence, we have

$$\begin{aligned} & Z_N \mathbb{P}_N(T_* \mu^N \in \mathcal{B}(\mu, \delta)) \\ &= \int \cdots \int_{\{z \in \mathcal{S}^N : T_* \mu^N \in \mathcal{B}(\mu, \delta)\}} \prod_{1 \leq i < j \leq N} |z_i - z_j|^\beta \prod_{i=1}^N (1 - |z_i|^2)^{\beta/2} e^{-N\mathbf{V}(z_i)} d\lambda(z_i). \end{aligned} \quad (2.11)$$

Then, with $F_{\mathbf{V}}$ defined in (2.5), one can write

$$\begin{aligned} & \prod_{1 \leq i < j \leq N} |z_i - z_j|^\beta \prod_{i=1}^N (1 - |z_i|^2)^{\beta/2} e^{-N\mathbf{V}(z_i)} d\lambda(z_i) \\ &= \exp \left\{ - \sum_{1 \leq i \neq j \leq N} F_{\mathbf{V}}(z_i, z_j) \right\} \prod_{i=1}^N (1 - |z_i|^2)^{\beta/2} e^{-\mathbf{V}(z_i)} d\lambda(z_i) \\ &= \exp \left\{ - N^2 \iint_{z \neq w} F_{\mathbf{V}}(z, w) dT_* \mu^N(z) dT_* \mu^N(w) \right\} \prod_{i=1}^N (1 - |z_i|^2)^{\beta/2} e^{-\mathbf{V}(z_i)} d\lambda(z_i). \end{aligned} \quad (2.12)$$

With $F_{\mathbf{V}}^M$ as in the proof of Proposition 2.3 (a) above, we have

$$\iint_{z \neq w} F_{\mathbf{V}}(z, w) dT_* \mu^N(z) dT_* \mu^N(w) \geq \iint_{z \neq w} F_{\mathbf{V}}^M(z, w) dT_* \mu^N(z) dT_* \mu^N(w). \quad (2.13)$$

Moreover, since \mathbb{P}_N -almost surely

$$T_* \mu^N \otimes T_* \mu^N(\{(x, y) \in \mathcal{S} \times \mathcal{S} : x = y\}) = \frac{1}{N},$$

we obtain conditionally to $T_*\mu^N \in \mathcal{B}(\mu, \delta)$

$$\begin{aligned}
& \iint_{z \neq w} F_{\mathbf{V}}^M(z, w) dT_*\mu^N(z) dT_*\mu^N(w) \\
& \geq \iint F_{\mathbf{V}}^M(z, w) dT_*\mu^N(z) dT_*\mu^N(w) - \frac{1}{N} \max_{\mathcal{S} \times \mathcal{S}} F_{\mathbf{V}}^M \\
& \geq \inf_{\nu \in \mathcal{B}(\mu, \delta)} \iint F_{\mathbf{V}}^M(z, w) d\nu(z) d\nu(w) - \frac{1}{N} \max_{\mathcal{S} \times \mathcal{S}} F_{\mathbf{V}}^M.
\end{aligned} \tag{2.14}$$

From (2.11)–(2.14) we find

$$\begin{aligned}
& \log \left\{ Z_N \mathbb{P}_N \left(T_*\mu^N \in \mathcal{B}(\mu, \delta) \right) \right\} \\
& \leq -N^2 \inf_{\nu \in \mathcal{B}(\mu, \delta)} \iint F_{\mathbf{V}}^M(z, w) d\nu(z) d\nu(w) \\
& \quad + N \left(\max_{\mathcal{S} \times \mathcal{S}} F_{\mathbf{V}}^M + \log \int_{\mathcal{S}} (1 - |z|^2)^{\beta/2} e^{-\mathbf{V}(z)} d\lambda(z) \right).
\end{aligned} \tag{2.15}$$

Note that by performing the change of variables $z = T(x)$ and using the growth assumption (1.7) it follows that

$$\int_{\mathcal{S}} (1 - |z|^2)^{\beta/2} e^{-\mathbf{V}(z)} d\lambda(z) = \int_{\mathbb{R}} e^{-V(x)} dx < +\infty,$$

and thus (2.15) yields

$$\limsup_{N \rightarrow \infty} \frac{1}{N^2} \log \left\{ Z_N \mathbb{P}_N \left(T_*\mu^N \in \mathcal{B}(\mu, \delta) \right) \right\} \leq - \inf_{\nu \in \mathcal{B}(\mu, \delta)} \iint F_{\mathbf{V}}^M(z, w) d\nu(z) d\nu(w). \tag{2.16}$$

The continuity of the map

$$\nu \mapsto \iint F_{\mathbf{V}}^M(z, w) d\nu(z) d\nu(w)$$

provides by letting $\delta \rightarrow 0$ in (2.16)

$$\limsup_{\delta \rightarrow 0} \limsup_{N \rightarrow \infty} \frac{1}{N^2} \log \left\{ Z_N \mathbb{P}_N \left(T_*\mu^N \in \mathcal{B}(\mu, \delta) \right) \right\} \leq - \iint F_{\mathbf{V}}^M(z, w) d\mu(z) d\mu(w), \tag{2.17}$$

and (c) is finally deduced by monotone convergence letting $M \rightarrow \infty$ in (2.17). \square

Equipped with Proposition 2.3, we are now in position to prove Theorem 1.1 thanks to the compactification procedure described in Section 2.1.

2.3 Proof of Theorem 1.1

Proof of Theorem 1.1. (a) Since $I_{\mathbf{V}}(\mu) = +\infty$ for all $\mu \in \mathcal{M}_1(\mathcal{S})$ such that $\mu(\{\infty\}) > 0$, we obtain from Lemma 2.1 and (2.7) that the levels sets of I_V and $I_{\mathbf{V}}$ are homeomorphic, namely for any $\alpha \in \mathbb{R}$

$$T_* \left\{ \mu \in \mathcal{M}_1(\mathbb{R}) : I_V(\mu) \leq \alpha \right\} = \left\{ \mu \in \mathcal{M}_1(\mathcal{S}) : I_{\mathbf{V}}(\mu) \leq \alpha \right\}.$$

Thus, Theorem 1.1 (a) follows from Proposition 2.3 (a).

(b) Theorem 1.1 (a) yields the existence of minimizers for I_V on $\mathcal{M}_1(\mathbb{R})$. Since T_* is a linear injection, it follows from (2.7) and Proposition 2.3 (b) that I_V is strictly convex on the set where it is finite, which warrants the uniqueness of the minimizer.

(c),(d) It is enough to show that for any closed set $\mathcal{F} \subset \mathcal{M}_1(\mathbb{R})$,

$$\limsup_{N \rightarrow \infty} \frac{1}{N^2} \log \left\{ Z_N \mathbb{P}_N \left(\mu^N \in \mathcal{F} \right) \right\} \leq - \inf_{\mu \in \mathcal{F}} I_V(\mu), \quad (2.18)$$

and for any open set $\mathcal{O} \subset \mathcal{M}_1(\mathbb{R})$,

$$\liminf_{N \rightarrow \infty} \frac{1}{N^2} \log \left\{ Z_N \mathbb{P}_N \left(\mu^N \in \mathcal{O} \right) \right\} \geq - \inf_{\mu \in \mathcal{O}} I_V(\mu). \quad (2.19)$$

Indeed, by taking $\mathcal{F} = \mathcal{O} = \mathcal{M}_1(\mathbb{R})$ in (2.18) and (2.19), one obtains

$$\lim_{N \rightarrow \infty} \frac{1}{N^2} \log Z_N = - \inf_{\mu \in \mathcal{M}_1(\mathbb{R})} I_V(\mu) = -I_V(\mu^*),$$

the latter quantity being finite.

Let us first show (2.18). We have for any closed set $\mathcal{F} \subset \mathcal{M}_1(\mathbb{R})$ that

$$\mathbb{P}_N \left(\mu^N \in \mathcal{F} \right) \leq \mathbb{P}_N \left(T_* \mu^N \in \text{clo}(T_* \mathcal{F}) \right), \quad (2.20)$$

where $\text{clo}(T_* \mathcal{F})$ stands for the closure of $T_* \mathcal{F}$ in $\mathcal{M}_1(\mathcal{S})$. Then, since $\mathcal{M}_1(\mathcal{S})$ is compact, so is $\text{clo}(T_* \mathcal{F})$, and by extracting a finite covering of $\text{clo}(T_* \mathcal{F})$ from an appropriate covering by balls, a classical argument from LDPs theory (see for example the proof of [DZ98, Theorem 4.1.11]) yields from Proposition 2.3 (c) that

$$\limsup_{N \rightarrow \infty} \frac{1}{N^2} \log \left\{ Z_N \mathbb{P}_N \left(T_* \mu^N \in \text{clo}(T_* \mathcal{F}) \right) \right\} \leq - \inf_{\mu \in \text{clo}(T_* \mathcal{F})} I_{\mathcal{V}}(\mu). \quad (2.21)$$

If $\nu \in \text{clo}(T_* \mathcal{F})$, then either $\nu \in T_* \mathcal{F}$ or $\nu(\{\infty\}) > 0$. Indeed, let $(T_* \eta_N)_N$ a sequence in $T_* \mathcal{F}$ with limit ν satisfying $\nu(\{\infty\}) = 0$. Lemma 2.1 yields $\eta \in \mathcal{M}_1(\mathbb{R})$ such that $\nu = T_* \eta$ and moreover the convergence of $(\eta_N)_N$ towards η . Since \mathcal{F} is closed, necessarily $\nu \in T_* \mathcal{F}$. As a consequence, since $I_{\mathcal{V}}(\mu) = +\infty$ as soon as $\mu(\{\infty\}) > 0$, we obtain from (2.7)

$$\inf_{\mu \in \text{clo}(T_* \mathcal{F})} I_{\mathcal{V}}(\mu) = \inf_{\mu \in T_* \mathcal{F}} I_{\mathcal{V}}(\mu) = \inf_{\mu \in \mathcal{F}} I_V(\mu). \quad (2.22)$$

Finally, (2.18) follows from (2.20)–(2.22).

We now prove (2.19). It is sufficient to show that for any $\mu \in \mathcal{M}_1(\mathbb{R})$ and any neighborhood $\mathcal{G} \subset \mathcal{M}_1(\mathbb{R})$ of μ we have

$$\liminf_{N \rightarrow \infty} \frac{1}{N^2} \log \left\{ Z_N \mathbb{P}_N \left(\mu^N \in \mathcal{G} \right) \right\} \geq -I_V(\mu). \quad (2.23)$$

For any k large enough, define $\mu_k \in \mathcal{M}_1(\mathbb{R})$ to be the normalized restriction of μ to the compact $[-k, k]$. Then $(\mu_k)_k$ converges towards μ as $k \rightarrow \infty$ and one moreover obtain by monotone convergence that

$$\lim_{k \rightarrow \infty} I_V(\mu_k) = I_V(\mu).$$

As a consequence, it is enough to show (2.23) under the extra assumption that the μ 's are compactly supported, so that the statement (2.19) is independent of the growth assumption on V . Thus, one can reproduce the proof of [AGZ10, Theorem 2.6.1] to show (2.23). The prove of Theorem 1.1 is therefore complete. \square

3 Generalizations

In this section we consider some generalizations of the result and the method presented in the previous sections.

3.1 Concerning the support of the Coulomb gaz

One could replace \mathbb{R} , that is the set where the particles x_i 's are distributed, by a closed subset $\Delta \subset \mathbb{C}$ with positive capacity, and (1.1) still makes sense. By positive capacity we mean that there exists $\mu \in \mathcal{M}_1(\Delta)$ with finite logarithmic energy $I(\mu) < +\infty$, see (2.9). We would then use the inverse stereographic projection from \mathbb{C} to the Riemann sphere, and a similar compactification procedure as described in Section 2.1 can be achieved (see [HK11] for more details). Then, parts (a), (b) and (c) of Theorem 1.1, and thus Corollary 1.2, hold without any substantial change of their proofs. Note that our proofs also cover the case where V is less regular, namely $V : \Delta \rightarrow \mathbb{R} \cup \{+\infty\}$ is only lower semi-continuous so that the set $\{x \in \Delta : V(x) < +\infty\}$ has positive capacity. Concerning Theorem 1.1 (d), if we still assume V to be continuous, one can easily perform an approximation procedure similarly to the one in the proof of Theorem 1.1 (d) to match with the setting presented in [Bl09], so that a full LDP holds.

3.2 Concerning the compactification procedure

Most of our proofs concerning the compactification procedure fit with a more general setting for which we summarize the consequences now.

Let \mathcal{X} be a locally compact, but not compact, Polish space and consider a sequence $(\mu^N)_N$ of random variables taking values in the space $\mathcal{M}_1(\mathcal{X})$ of Borel probability measures on \mathcal{X} . Let $(\hat{\mathcal{X}}, T)$ be an Alexandrov compactification of \mathcal{X} , that is a compact set $\hat{\mathcal{X}}$ with an element $\infty \in \hat{\mathcal{X}}$ such that $T : \mathcal{X} \rightarrow \hat{\mathcal{X}}$ is a homeomorphism on its image $T(\mathcal{X})$ and $\hat{\mathcal{X}} \setminus T(\mathcal{X}) = \{\infty\}$. Define T_* to be the push-forward by T similarly as in (2.1). We equip $\mathcal{M}_1(\hat{\mathcal{X}})$ with its weak topology, so that it becomes a compact Polish space, and denotes $\mathcal{B}(\mu, \delta)$ the ball centered in $\mu \in \mathcal{M}_1(\hat{\mathcal{X}})$ with radius $\delta > 0$.

Let $(\alpha_N)_N$ and $(Z_N)_N$ be two sequences of real positive numbers with $\lim_{N \rightarrow \infty} \alpha_N = +\infty$. Assume there exists a map $\Phi : \mathcal{M}_1(\hat{\mathcal{X}}) \rightarrow \mathbb{R} \cup \{+\infty\}$ which satisfies the following.

- (a) For all $\mu \in \mathcal{M}_1(\hat{\mathcal{X}})$, $\Phi(\mu) = +\infty$ as soon as $\mu(\{\infty\}) > 0$.
- (b) For all $\mu \in \mathcal{M}_1(\hat{\mathcal{X}})$,

$$\limsup_{\delta \rightarrow 0} \limsup_{N \rightarrow \infty} \frac{1}{\alpha_N} \log \left\{ Z_N \mathbb{P}_N \left(T_* \mu^N \in \mathcal{B}(\mu, \delta) \right) \right\} \leq -\Phi(\mu).$$

Then for any closed set $\mathcal{F} \subset \mathcal{M}_1(\mathcal{X})$,

$$\limsup_{N \rightarrow \infty} \frac{1}{\alpha_N} \log \left\{ Z_N \mathbb{P}_N \left(\mu^N \in \mathcal{F} \right) \right\} \leq - \inf_{\mu \in \mathcal{F}} \Phi \circ T_*(\mu).$$

Moreover, Φ has compact level sets (resp. is strictly convex on the set where it is finite) if and only if $\Phi \circ T_*$ has (resp. is).

A similar strategy is used in [HK] where a LDP is established for a multi-type particles 2D Coulomb gas related to an additive perturbation of a Wishart random matrix model.

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References

- [AGZ10] G. W. Anderson, A. Guionnet and O. Zeitouni, *An introduction to random matrices*, Cambridge Studies in Advanced Mathematics Vol. 118, Cambridge University Press, Cambridge, England (2010).
- [AN07] R. Ash and W. Novinger, *Complex variables*, Dover publication, Second edition (2007).
- [Bl09] T. Bloom, *Voiculescu's entropy and potential theory*, Annales de la faculté des sciences de Toulouse Sér. 6, 20 no. S2: Special issue: Proceedings of the Symposium Analyse Complexe et Applications in Honor of Nguyen Than Van (2011) 57–69.
- [BAG97] G. Ben Arous and A. Guionnet, *Large deviations for Wigner's law and Voiculescu's non-commutative entropy*, Probab. Theory Rel. Fields 108 (1997) 51–542.
- [CKL98] U. Cegrell, S. Kolodziej, and N. Levenberg, *Two problems on potential theory with unbounded sets*, Math. Scand. 83 (1998) 265–276.
- [D02] R.M. Dudley, *Real analysis and probability*, Cambridge Studies in Advanced Mathematics Vol. 74, Cambridge University Press, Cambridge, England (2002).
- [DG09] P. Deift and D. Gioev, *Random matrix theory : invariant ensembles and universality*, Courant Lecture Notes in Mathematics Vol. 18, AMS Providence R.I. (2009).
- [DZ98] A. Dembo and O. Zeitouni, *Large deviations techniques and applications*, second edition, Springer, New York, NY (1998).
- [HK] A. Hardy and A. B. J. Kuijlaars, *Large deviations for a non-centered Wishart ensemble*, Work in preparation.
- [HK11] A. Hardy and A. B. J. Kuijlaars, *Weakly admissible vector equilibrium problems*, Preprint arXiv:1110.6800 (2011).
- [HP00] F. Hiai and D. Petz, *The semicircle law, free random variables and entropy*, AMS Providence R.I. (2000)